Calculus

Limits, Derivatives and Integrals

XYZW Project

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Preface

Calculus answers two questions: how fast a quantity is changing, and how large an accumulation of many tiny quantities becomes. Both answers rest on a single idea — the limit.

Route

1. Limits: turn "arbitrarily close" into inequalities you can check.
2. Derivatives: define the instantaneous rate of change with a limit.

Prerequisites

You should be comfortable with real numbers, functions and inequalities. Linear algebra is not required, but Vectors from *Linear Algebra* makes the multivariable parts easier.

Conventions

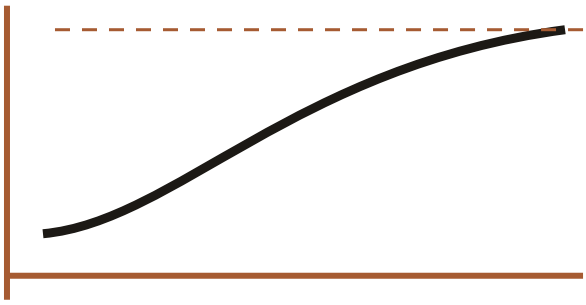
Functions are real-valued and defined on an interval of $\mathbb{R}$ unless stated otherwise. Differentials are always written $\mathrm{d}x$.

Chapter 1

Limits

1.1 Intuition

When $f(x)$ gets closer and closer to a number L as x approaches a, we call L the limit of f at a. The dashed line in the figure is that L.



1.2 The precise definition

Limit

Let f be defined on a punctured neighbourhood of a. If for every $\varepsilon > 0$ there is a $\delta > 0$ such that $0 < |x - a| < \delta$ implies

$$|f(x) - L| < \varepsilon,$$

then L is the **limit** of f at a, written $\lim_{x \to a} f(x) = L$.

The value at the point does not matter

The definition only looks at $x \neq a$. Whether $f(a)$ is defined, and what it equals, has no effect on the limit.

1.3 An example

Prove that $\lim_{x \to 2}(3x + 1) = 7$

Given $\varepsilon > 0$, choose $\delta = \varepsilon/3$. Whenever $0 < |x-2| < \delta$,

$$|(3x+1) - 7| = 3|x-2| < 3\delta = \varepsilon.$$

1.4 Limit laws

If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then

$$\lim_{x\to a} (f(x) + g(x)) = L + M, \qquad \lim_{x\to a} f(x)\, g(x) = LM.$$

With limits in hand, Derivatives can be defined.

Chapter 2

Derivatives

2.1 Definition

Derivative

If the limit

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists, f is **differentiable** at a and $f'(a)$ is its **derivative** there.

Geometrically, $f'(a)$ is the slope of the tangent line at $(a, f(a))$. The notion of limit used here is the one from Limits › The precise definition.

2.2 Common derivatives

$f(x)$	$f'(x)$
x^n	nx^{n-1}
e^x	e^x
$\ln x$	$1/x$
$\sin x$	$\cos x$

2.3 A numerical check

For small h the difference quotient approaches the derivative. This program uses a central difference to check that $\sin' = \cos$:

```ts
const derivative = (f: (x: number) => number, x: number, h = 1e-5): number =>
  (f(x + h) - f(x - h)) / (2 * h)

console.log(derivative(Math.sin, 0)) // ≈ 1 = cos(0)
```

Why a central difference

The central difference has error $O(h^2)$; a one-sided difference has error $O(h)$. For the same h, the former is far more accurate.

2.4 The chain rule

If g is differentiable at a and f is differentiable at $g(a)$, then

$$\frac{\mathrm{d}}{\mathrm{d}x} f(g(x))\Big|_{x=a} = f'(g(a))\, g'(a).$$