Linear Algebra

Vectors, Matrices and Linear Maps

XYZW Project

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Preface

Linear algebra studies objects that can be added and scaled, together with the maps that respect those two operations. It is the shared language of geometry, calculus, probability, statistics and machine learning.

How to read this book

Every chapter unfolds in the same order: a picture, then a definition, then examples. Feel free to skip proofs on a first reading, but do not skip the examples.

> **Draw it first**
>
> When you meet a new definition, draw it in the plane $\mathbb{R}^2$. Most statements that make sense in two dimensions generalize to $\mathbb{R}^n$ by notation alone.

Notation

Symbol	Meaning
$\mathbb{R}^n$	the real vector space of dimension n
u, v, w	vectors
A, B	matrices
$\|v\|$	the length of the vector v

Vectors are written as columns[1], for example

$$v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

How this book relates to the rest of the series

The rigorous definition of a limit lives in Limits of *Calculus*. Linearity of expectation (see Random Variables › Linearity of expectation) says, at heart, that expectation is a linear map.

When you are ready, start with Vectors.

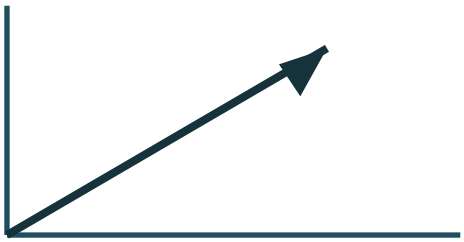
[1] Some books use row vectors. The two conventions differ by a single transpose.

Chapter 1

Vectors

1.1 The geometric picture

In the plane, a vector is an arrow with a direction. Placing two arrows head to tail adds them; stretching, shrinking or flipping an arrow multiplies it by a scalar.



1.2 Vector spaces

Vector space

Let V be a set with an addition $V \times V \to V$ and a scalar multiplication $\mathbb{R} \times V \to V$. If commutativity, associativity, a zero, negatives and the distributive laws hold for all $u, v, w \in V$ and $a, b \in \mathbb{R}$, then V is a **vector space** over $\mathbb{R}$.

The most common example is $\mathbb{R}^n$: add componentwise and multiply every component by the same number.

1.3 Linear combinations

Linear combination

Given vectors $v_1, \dots, v_k$ and numbers $a_1, \dots, a_k$, the vector

$$w = a_1 v_1 + a_2 v_2 + \cdots + a_k v_k$$

is a **linear combination** of $v_1, \dots, v_k$.

The set of all linear combinations is the **span** of the vectors, written $\operatorname{span}(v_1, \dots, v_k)$.

1.4 Length and inner product

The length of a vector in $\mathbb{R}^n$ comes from Pythagoras:

$$\|v\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

The inner product $\langle v, w \rangle = \sum_{i=1}^{n} v_i w_i$ satisfies $\langle v, v \rangle = \|v\|^2$.

Check it in code

```python
import math

def norm(v: list[float]) -> float:
    return math.sqrt(sum(x * x for x in v))

print(norm([3.0, 4.0]))  # 5.0
```

The next chapter writes linear combinations more compactly with matrices; see Matrices › Matrix multiplication.

Chapter 2

Matrices

2.1 Bookkeeping for linear combinations

Arrange vectors $v_1, \dots, v_n \in \mathbb{R}^m$ as columns to get the $m \times n$ matrix $A = (v_1 \ \cdots \ v_n)$. A linear combination then becomes a matrix–vector product:

$$Ax = x_1 v_1 + \cdots + x_n v_n.$$

This is just another way to write a linear combination.

2.2 Matrix multiplication

Matrix multiplication

Let A be an $m \times n$ matrix and B an $n \times p$ matrix. The product $C = AB$ is the $m \times p$ matrix with entries

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

For example,

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 11 \\ 4 \end{pmatrix}.$$

Order matters

In general $AB \neq BA$. Even when AB is defined, BA may not be defined at all.

2.3 Associativity

Associativity

Whenever the products are defined, $(AB)C = A(BC)$.

Proof

Compare the (i, j) entries of both sides:

$$((AB)C)_{ij} = \sum_l \Big(\sum_k a_{ik} b_{kl} \Big) c_{lj} = \sum_k a_{ik} \Big(\sum_l b_{kl} c_{lj} \Big) = (A(BC))_{ij}.$$

The middle step only swaps the order of two finite sums.

2.4 Common matrices

Name	Condition	Example
Identity I	$AI = IA = A$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
Diagonal	off-diagonal entries are 0	$\mathrm{diag}(2, 3)$
Symmetric	$A^\top = A$	$\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$